ABDUCTIVE LOGICS IN A BELIEF REVISION FRAMEWORK

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August 2002

Abstract

Abduction was first introduced by the philosopher C.S. Peirce in the epistemological context of scientific discovery. It was more recently analyzed in Artificial Intelligence, especially with respect to diagnosis analysis or ordinary reasoning. These two fields share a common view of abduction as a general process of hypotheses formation. More precisely, abduction is conceived as a kind of reverse explanation where a hypothesis H can be abduced from facts E if H is a "good explanation" of E. The paper aims at giving a general logical picture of abduction which could be used in both fields.

The most standard way to define a good explanation is through deduction, "classical abduction" from E to H being then defined as reverse deduction: $H \subseteq E$. Since such a definition can be shown to be unsatisfactory, a richer approach consists in introducing moreover a belief revision operation. In a semantic belief revision framework, an agent's initial belief K is revised into a final belief K*A when the agent receives some message A. Replacing H or E by the respective beliefs K*H or K*E leads naturally to three possible alternative schemes to reverse deduction.
The paper studies these alternative schemes which are first evaluated through the intuitive relevance of their semantic definitions, considering the general heuristic that an abduction must be the reverse of a good explanation. Some examples which appear intuitively to be desirable or not are given to support our argumentation. Second, sets of axioms for the three alternatives and proofs for the corresponding representation theorems are given. Third, the three alternatives are compared through the more or less strong axioms on which they rest. Finally, on semantic grounds as well as on axiomatic grounds, one abduction definition which was never vindicated by previous work is selected. This definition, named "ordered abduction" says that H is abduced from E if and only if $K*H \subseteq K*E$. It leads to consider abduction as a logical relation which cannot be directly defined by the inversion of a consequence relation (either deductive or non monotonic).

Keywords: Abduction; Belief revision; Explanation; Non monotonic reasoning.

1. INTRODUCTION

Initially, abduction was defined in **epistemology** as a reasoning process leading to form an explanatory hypothesis from given observations, especially in physics. It operates from facts to facts, for instance when Leverrier postulated the existence of Neptune from the discrepancy between the predicted and the observed trajectory of Uranus. It operates from facts to laws, for instance when the law of discrete electromagnetic rays was derived from observations of different chemical elements. It operates from laws to theory, for instance when Newton's theory was conjectured from Kepler's laws and the falling bodies law.

More recently, instances of abduction were given in **Artificial Intelligence (AI)**, especially in relation with diagnosis tasks or ordinary reasoning. The first are illustrated by medical diagnosis when a physician guesses the illness which causes some symptoms or by police inquiry when a police officer guesses a criminal from observed clues. The second are found in natural interpretation when an agent tries to reveal his opponent's preferences (or beliefs) through his actions, or in experimental psychology when people try to discover a recurrence rule able to generate a given sequence of numbers.

The aim of the paper is to propose a general definition that suits all the typical instances of abduction as a process of hypotheses formation, either in science or in diagnosis or in ordinary reasoning. The concept of abduction was first defined by the philosopher Charles Peirce in the following terms: "Abduction is the process of forming an explanatory hypothesis. It is the only logical operation which introduces any new idea." He gave in fact two rather different definitions, a formal one introduced in the treatment of a syllogism and a constructive one stated in the process of belief formation. However, the common idea is to consider abduction as some kind of reverse explanation, in that a proposition abduced from another one must be a good explanation for it.

More precise logical definitions need to be given. A first candidate is to reduce explanation to deduction and to consider what is called "classical abduction" as inverse deduction. But such a definition appears as unsatisfactory since it retains abnormal explanations grounded on implausible assumptions while excluding relevant explanations which accept some exceptions. More sophisticated definitions rely on belief revision operations since some kind of explanation, i.e. non monotonic reasoning, was already associated to belief revision. But again, abduction cannot be interpreted simply as reverse belief revision and more elaborated links with belief revision have to be proposed.

The paper introduces four abduction schemes (including classical abduction), reciprocal of four explanation schemes, each couple being defined by a logical relation between two belief revision operations. Further on, each form of abduction will be defined by an axiom system and justified by a representation theorem linking this axiom system and its semantic definition. One abduction scheme (ordered abduction) will be favored, first according to its relevance for the motto of inference to a good explanation, second according to the axioms it follows.

The paper is organized as follows. The second section recalls the historical background and introduces the formal framework which will be used. The third section defines the four possible abduction schemes in relation with belief revision operations. The fourth section compares the relevance of these schemes through an example and more theoretical considerations and considers the related works. The fifth section presents the axiom systems for non transitive, non reflexive and ordered abduction and gives representation theorems for the two last schemes. The sixth section compares these axiom systems and discusses their respective advantages and defaults. A conclusion follows while proofs are given in appendix.

2. BACKGROUND AND FRAMEWORK

2.1. Abduction along Peirce

The first definition of abduction given by Peirce (1931-1958), **abduction$_1$**, stands inside the predicate calculus framework. Consider a syllogism which relates a structural antecedent H (the rule) and a factual antecedent h (the case) to a factual consequent k (the result): $H \wedge h \rightarrow k$. According to Peirce, there are three basic operations between these terms:

- **prediction** links H and h to k
- **abduction$_1$** links H and k to h
- **induction** links couples (h,k) to H

This analysis is in accordance with the so called "deductive-nomological scheme", on which Hempel (1965) and Popper (1959) relied for building their epistemological theories. Popper put stress on **refutation**, which links $\neg$k to $\neg$H or $\neg$h. Hempel proposed a theory of **confirmation**, a concept which encompasses both abduction$_1$ and induction. The two last concepts appear technically as reverse predictions, although induction selects inference to rules (in the context of a case) and abduction$_1$ selects inference to cases (in the context of a rule). But contrary to deduction which preserves the truth value of propositions, abduction (like induction) cannot be logically justified and even falls apparently in the fallacy called "the affirmation of the consequent". Actually, abduction is knowledge ampliative.

Although precise, this first definition of abduction encounters two obvious limits:
- some valid abductions are intuitively not admissible because cases of some abnormal rules are not excluded by this scheme. For instance, if I see that my grass is wet, I would generally not assume that a water bomber has poured the content of its tank on it.
- some intuitively admissible abductions are not valid because they rely on non nomological relations between a fact and a possible explanation. For instance, if I see that my grass is wet, I cannot abduce that my sprinkler is on since the sprinkler may fail.
Furthermore, the clear logical distinction between abduction and induction could be considered as too restrictive, abduction$_1$ preventing any form of inference from facts to laws or to theories to be called abduction.

Always according to Peirce, **abduction$_2$** is a more general mode of inference which is defined in the dynamic context of scientific inquiry. A scientist may learn a surprising fact, that troubles his mental state of "cognitive calm" concerning a given class of phenomena. This surprising fact requires an explanation validated in three reasoning steps:

- abduction$_2$ corresponds to a first step where the scientist formulates some explanatory hypotheses (laws or theories) which, if true, would restore his state of "cognitive calm";
- deduction corresponds to a second step where the scientist infers from the preceding hypotheses some contrasted consequences able to be experimentally tested;

- induction corresponds to a third step where the scientist proceeds to experiments in order to build degrees of confirmation of the hypotheses, leading eventually to favor one.

A possible reading of this theory is that abduction$_2$ belongs to the context of discovery, the context of justification being reserved to deduction and induction. This could imply that a logical analysis of abduction$_2$ is impossible since heuristics is not a purely logical process. Furthermore, even if logification is relevant, abduction$_2$ would not even be an inference because it does not lead to "conclusions" but to mere "candidates to belief". However, according to most Peirce's analysts, a logic of abduction$_2$ can be proposed since not every hypothesis is admissible as a good candidate for belief: even if not accepted, abduced hypotheses result from a selection of the explanations that can be "seriously considered" for further acceptance. This requires to propose a logical criterion for this selection.

Actually, abduction$_2$ is not incompatible with abduction$_1$. It can rather be thought of as a more general inference which associates abduction with two constraints:

(i) abduced hypotheses must "explain" the facts under consideration, eventually in a given context;

(ii) abduced hypotheses must be "good candidates to belief".

A motto which seems to encompass both constraints and is often endorsed by abduction theorists is that abduction is **inference to the best explanation** (see Harman, 1978, Thagard, 1978, Lipton, 1991 for a detailed analysis of this concept and van Fraassen, 1980, for a critical appraisal of its use in favor of scientific realism). However, the notion of "the best" explanation is too demanding since abduction may select several candidates to belief. Hence, the guideline for a further analysis will be that abduction is simply **inference to a good explanation**. Usually, an explanation scheme appears as a "forward inference" which involves an explanans A (for instance a case) explaining an explanandum B (for instance a result), eventually in some context (for instance a law). Conversely, an abduction scheme can be viewed as a "backward inference" from the explanandum B to the explanans A, a condition realized by both abduction$_1$ and abduction$_2$.

2.2. Belief revision

Belief revision theory can be modeled in two alternative logical frameworks. The syntactic framework is defined by a formal language **L** built by use of a finite set of propositions {a, b, …} closed under the connectives $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction) and $\rightarrow$ (implication). Let T and $\perp$ be the two constants truth and falsity. Let $\vdash$ be the symbol of the meta-level deduction operation. The semantic (set-theoretic) framework is defined on a finite set of possible worlds with the set operations - (complementation), $\cap$ (intersection), $\cup$ (union) and $\subseteq$ (inclusion). Let A, B, … be subsets (or events), characterized by the fact that respectively a, b, … are true in each of the worlds inside them and are false in each of the worlds outside them. Let W and $\varnothing$ be respectively the full set and the empty set.

The two frameworks are isomorphic under standard conditions with the following correspondence between symbols: $\neg$, $\cap$, $\cup$, $\subseteq$ for -, $\wedge$, $\vee$, $\vdash$. In the following, we will use the set-theoretic framework.

Belief revision is a belief change operation which relates an initial agent's belief K (on a static universe) and a message A (which may contradict the initial belief) to a final belief K*A. Beliefs K and K*A are assumed to be subsets of W. Contrary to W, K is assumed to evolve when the agent makes new observations or receives new informations from other agents. The basic postulate of belief revision is that the message has an epistemic priority over the initial belief of an agent, due to more direct observations or more reliable sources. This postulate is shared by abductive reasoning.

In a lot of AI works, it is usual to introduce explicitly a *background theory* Σ. Such a theory considers some generic beliefs endorsed by the agent. In the belief revision framework, such a background theory will be considered as embedded partially in W and partially in K. If an element of Σ is fixed, it is directly incorporated as a constraint in the set W. If an element of Σ may change, it is included in the belief K of the agent, which contains generic beliefs (i.e. rules) as well as specific ones (i.e. facts).

Belief revision was duly axiomatized by Alchourron, Gärdenfors & Makinson (1985) according to the following axioms:

A1. Consistency
If $K \neq \emptyset$ and $A \neq \emptyset$ then $K*A \neq \emptyset$

A2. Success
$K*A \subseteq A$

A3. Conservation
If $K \subseteq A$ then $K*A = K$

A4. Sub-Expansion
$(K*A) \cap B \subseteq K*(A \cap B)$

A4'. Inclusion
$K \cap A \subseteq K*A$

A5. Super-Expansion
If $(K*A) \cap B \neq \emptyset$ then $K*(A \cap B) \subseteq (K*A) \cap B$

A5'. Preservation
If $K \cap A \neq \emptyset$ then $K*A \subseteq K \cap A$

A45. Right Distributivity
$K*(A \cup B) = K*(A) \cup K*(B)$ or $K*A$ or $K*B$

The basic axiom system is **A =** {A1, A2, A3, A4, A5}.

It is possible to prove the following theorems:
a) Under A2, A45 is equivalent to {A4, A5}.
b) {A3, A4} implies A4' and{A3, A5} implies A5'. Actually A3 is not needed but only $K*T=K$.

c) {K*T=K, A4} implies A4' and {K*T=K, A5} implies A5'.
d) {A4', A5'} implies A3

Hence **A** is equivalent to the following axiom systems:
- {A1, A2, A4, A5, (K*T=K)}
- {A1, A2, A4, A4', A5, A5'}
- {A1, A2, A3, A45}
- {A1, A2, A45, (K*T=K)}

Belief revision rules can be associated to the axiom system by a representation theorem (Alchourron, Gärdenfors & Makinson, 1985). Consider a preference relation $<_K$ (and an associated equivalence relation $=_K$) on W indexed on a subset K of W. This preference relation is assumed to be a total preorder and to fulfill two properties:

(i) $w' \in K$ and $w'' \in K \Rightarrow w' =_K w''$

(ii) $w' \in K$ and $w'' \notin K \Rightarrow w' <_K w''$

It defines a ranking of the worlds of W, which can be represented by a system of concentric "spheres" around K. These embedded spheres cut up coronas between two successive ones. The most distant coronas correspond to the subsets of the least preferred worlds. The minimal worlds of an event A (called the 'preferred' or the 'normal' part of A) are now defined by:

$\mathrm{Min}_K(A) = \{w \in A: \forall w' \in A \;\; w' <_K w \text{ is false}\}$

The representation theorem states that the revised belief is the set of the minimal worlds belonging to the message (the "preferred" part of the message):

$K{*}A = \mathrm{Min}_K(A)$

It means that the final belief is the intersection between A and the sphere of the closest worlds to K which has a non empty intersection with A (see figure 1).

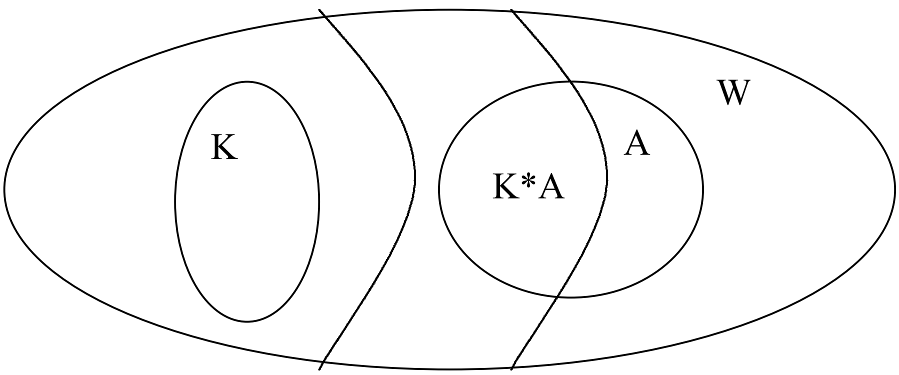

Figure 1

The preference relation $<_K$, which is specific of one agent's "epistemic state" (Darwiche and Pearl (1997), is a more complete description of the agent's total belief than K and defines all what is needed to achieve his belief revision process.

2.3. Non monotonic reasoning

Non monotonic inference weakens the usual operation of deduction in order to reflect rules of common reasoning in the context of proof. These rules do not preserve anymore the truth value of the propositions. A non monotonic inference A |~ B is an inference which states that: "if A, normally B" or "if A is considered as true, then B is accepted". This kind of inference is non monotonic since adding a new premise A' to A does not necessarily preserve the initial conclusion B.
Non monotonic inference was duly axiomatized by Kraus, Lehmann & Magidor (1990) according to the following axioms (see Lehmann & Magidor, 1992):

C0. Left Logical Equivalence
If A ≡ B and A |~ C then B |~ C

C1. Right Weakening
If A ⊆ B and C |~ A then C |~ B

C2. Reflexivity
A |~ A

C3. Right And
If A |~ B and A |~ C then A |~ B ∩ C

C4. Left Or
If A |~ C and B |~ C then A U B |~ C

C5'. Cautious Monotony.
If A |~ B and A |~ C then A ∩ B |~ C

C5. Rational Monotony
If (not (A |~ -B) and A |~ C) then A ∩ B |~ C
(axiom C5' is a weakening of axiom C5)

C6. Supra Classicality
If A |- B then A |~ B

C7. Conditionalization
If A ∩ B |~ C then A |~ B→C

C8. Cut
If A |~ B and A ∩ B |~ C then A |~ C

The axiom system $\mathbf{C_p}$={C0, C1, C2, C3, C4, C5'} defines a **preferential** non monotonic inference. The axiom system $\mathbf{C_r}$={C0, C1, C2, C3, C4, C5, C5'} defines a **rational** non monotonic inference; it entails axioms C6, C7, C8.
Representation theorems were given (Kraus, Lehmann & Magidor, 1990; Lehmann & Magidor, 1992). Consider a preference relation defined by a strict partial order < on W such that the minimal worlds of an event A, denoted Min(A) are defined by:

$Min(A) = \{w \in A: \forall w' \in A, w' < w \text{ is false}\}$

Then a non monotonic inference $A \mid\sim B$ is characterized by the fact that the consequent can be deduced from the minimal worlds of the antecedent (the "preferred" or "normal" part of it):

$A \mid\sim B$ iff $Min(A) \subseteq B$

The different kinds of non monotonic logics can be defined according to the properties of the preference relation (Lehmann & Magidor, 1992). A **preferential** non monotonic inference is obtained when the strict partial order < on W is *smooth*, i.e. when: $\forall X \neq \emptyset \in 2^W$, $Min(X) \neq \emptyset$. A **rational** non monotonic inference is obtained when the strict partial order < on W is *modular*, i.e. such that: $\forall x, y, z \in W$, if $x < y$ then either $z < y$ or $x < z$ (which is equivalent to the fact that it is *negatively transitive*, i.e. that the complementary relation $\geq$ is transitive).

The following correspondence rule between rational non monotonic inference and belief revision has been proved (Gärdenfors & Makinson, 1991):

$A \mid\sim_K B$ iff $K^*A \subseteq B$

The initial belief K acts as a parameter for specifying partially the preference relation underlying the non monotonic inference:

$K = \cap B: T \mid\sim_K B$

3. FOUR ABDUCTION SCHEMES

3.1. Abduction as belief revision

Within epistemology, although the concept of explanation has been widely investigated, the debate on abduction itself has deserved less consideration and gained few major logical improvements since Peirce (however see Rescher, 1978, Levi, 1979). Within Artificial Intelligence, abduction enjoyed more recently a renewed popularity and a lot of papers have proposed many definitions and as many abductive logics. They are all defined in the syntactic (propositional) framework or in the associated semantic (possible worlds) framework. Some of them are explicitly stated or may be translated within the belief revision scheme (see section 4.3.).

Belief revision is more and more widely accepted as very powerful and convenient to model reasoning. First, belief revision theory can be stated in the possible worlds semantics which is very intuitive and allows to grasp easily the meanings of the axioms. Second, belief revision inference seems to be a very fundamental operation that can be linked with many different types of inference. It has been not only related to non monotonic reasoning (Kraus, Lehmann & Magidor, 1990), but also to confirmation (Zwirn & Zwirn, 1994) and (in an "updating context" with a dynamic universe rather than in a "revising context") to conditional reasoning (Stalnaker, 1968).

The problem considered in the paper is to examine how to link abduction to belief revision. The heuristic arguments for doing so are the following:

- abduction rests necessarily upon some belief change operation. It relates an observation which changes our initial belief (whether it contradicts it or not) to a hypothesis which is assumed to be a good explanation of this observation when considering the final belief;

- abduction is, as belief revision, ampliative and non monotonic. When a hypothesis is a good explanation of some facts, that doesn't mean that it is a good explanation of these facts jointly to some other facts.

However, one cannot consider that belief revision or non monotonic inference are directly relevant theories of abductive reasoning. Such a "direct equivalence" would state that a hypothesis H is abduced from facts E either iff K*E = H (belief revision) or iff E $|\!\sim$ H (non monotonic inference). Such a thesis has to be rejected for two reasons. First, this use of belief revision or of non monotonic reasoning introduces a direct inference from facts to hypotheses. However, as considered in this paper, abduced hypotheses have to be an explanation of facts and need to entail them in some way. Second, hypotheses implied by a belief revision or resulting from a non monotonic inference are "accepted" by the agent and integrated in his final belief. However, as considered in this paper, abduced hypotheses are only "serious candidates" for acceptation and their acceptance depends on further tests between them. Hence, a good logical definition of abduction must state which belief revision operations are adequately involved when selecting hypotheses which are "seriously considered" without being necessarily accepted.

In order to integrate abduction in the preceding semantic framework, it is convenient to add two symmetrical operators: $|\!\rightarrow$ and $\|\!\rightarrow$, with the following interpretation:

$H |\!\rightarrow E$: event E is (well) explained by the hypothesis H

$E \|\!\rightarrow H$: the hypothesis H is abduced from event E

The equivalence relation $E \|\!\rightarrow H$ iff $H |\!\rightarrow E$ holds by definition. The arrow $\rightarrow$ used above generically for all forms of abduction (or explanation) will be replaced hereafter by different signs for each specific type of abduction (or explanation) in order to relate easily the different schemes.

3.2. Formal definition of the abduction schemes

The basic schemes usually considered are **classical explanation** and **classical abduction** conjointly defined by the following definitions (where $\|\!-$ must not be interpreted as semantic deduction):

$H \vdash E$ iff $H \subseteq E$

$E \|\!- H$ iff $H \subseteq E$

(The label "classical" refers to classical logic where no belief revision operation is involved). This abduction scheme is the most straightforward conception of an inference to a good explanation. It will be enriched by replacing E or H by K*E or K*H in order to give rise to three other schemes.

The second couple of schemes defines respectively **non transitive explanation** and **non transitive abduction** by the following definitions:

$H |\!\sim E$ iff $K{*}H \subseteq E$

$E \|\!\sim H$ iff $K{*}H \subseteq E$

(The label "non transitive" is favored over the label "non monotonic" since other explanation and abduction schemes will be non monotonic while this one is the only one to be non transitive). This abduction scheme is logically weaker than the previous one (i.e. a hypothesis abduced with classical abduction will be also abduced with non transitive abduction). It considers that abduction is not reverse deduction but rather reverse belief revision (hence reverse non monotonic inference): one abduces a hypothesis from a fact if one would have added this fact to one's belief after having revised initial belief by the hypothesis (or equivalently if one infers non monotonically the fact from the hypothesis).

A third couple of schemes defines **non reflexive explanation** and **non reflexive abduction** by the following definitions (including for technical reasons, that a contradiction cannot be abduced):

$H \mid\!\prec E$ iff $H \subseteq K * E$

$E \,\|\!\prec H$ iff $\varnothing \neq H \subseteq K*E$

(This abduction scheme is called non reflexive since it is the only one which respects that property). It is logically stronger than classical abduction. It states that one abduces a hypothesis from a fact if this hypothesis explains deductively the revised fact.

The last couple of inferences is **ordered explanation** and **ordered abduction**, respectively defined by:

$H \mid\!\approx E$ iff $K * H \subseteq K * E$

$E \,\|\!\approx H$ iff $\varnothing \neq K*H \subseteq K*E$

(The term ordered has been chosen since the binary relation is now reflexive and transitive and hence it is a pre-order; it is the only abduction scheme satisfying these properties except for classical abduction). This abduction scheme is stronger than non transitive abduction, weaker than non reflexive abduction, and cannot be compared to classical abduction. It considers that antecedent and consequent are both contextualized by prior belief and relies on the fact that the belief revised by the hypothesis would logically imply the belief revised by the fact.

3.3. A synthetic table

In table 1, the four explanation operations are located around the center and the four corresponding abduction operations in the periphery. Moreover, the relations of implication between them are denoted in the following way:

- infra (resp.supra)-classicality means that the scheme is stronger (resp. weaker) than classical abduction
- infra (resp. supra)-ordinality means the scheme is stronger (resp.weaker) than ordered abduction

classical abduction $E \Vdash H$ iff $H \subseteq E$ *supra-* if $H \vdash E$ if $E \Vdash H$ **classical explanation** $H \vdash E$ iff $H \subseteq E$ *infra-classicality* if $H \mid\prec E$, then $H \vdash E$	**non transitive abduction** $E \Vert\!\sim H$ iff $K{*}H \subseteq E$ *classicality* then $H \mid\!\sim E$ then $E \Vert\!\sim H$ **non transitive explanation** $H \mid\!\sim E$ iff $K{*}H \subseteq E$ *supra-ordinality* if $H \mid\approx E$, then $H \mid\!\sim E$
if $E \Vert\prec H$, then $E \Vdash H$ **non reflexive explanation** $H \mid\prec E$ iff $H \subseteq K{*}E$ *infra-* if $H \mid\prec E$ if $E \Vert\prec H$, **non reflexive abduction** $E \Vert\prec H$ iff $H \subseteq K{*}E$	if $E \Vert\approx H$, then $E \Vert\!\sim H$ **ordered explanation** $H \mid\approx E$ iff $K{*}H \subseteq K{*}E$ *ordinality* then $H \mid\approx E$ then $E \Vert\approx H$ **ordered abduction** $E \Vert\approx H$ iff $K{*}H \subseteq K{*}E$

Table 1

4. SEMANTIC COMPARISON OF ABDUCTION SCHEMES

4.1. An example

A physician is confronted with a young urban child who has regular coughing fits. Let's consider the following sets of worlds:

E: the patient coughs (this is the fact to explain)

H_1: the patient has bronchitis
H_2: the patient has asthma
H_3: the patient has just made a big effort and has no disease
H_4: the patient has the sickness of the farmer's lung (a peculiar kind of inflammation of the lung's alveolus due to exposition to vegetal or animal dust)
H_5: the patient has no disease but he feigns coughing
H_6: the patient has no disease and doesn't feign anything
In this example, for the sake of simplicity, the possible reasons explaining E are limited to H_1, H_2, H_3, H_4, H_5, H_6 (assumed to be incompatible):

From the medical background theory and his experience, the physician knows that:
- coughing fits always appear with bronchitis and when the patient feigns coughing
- coughing fits usually appear with asthma and with the sickness of the farmer's lung but do not in exceptional cases
- coughing fits can appear or not in usual cases of big effort
- bronchitis and asthma are more usual than the sickness of the farmer's lung which is very rare
- it is rare for a patient to feign coughing but more frequent than to have the sickness of the farmer's lung
- during the first visit, most often the patient has no disease and comes for a routine visit

Let's call (X running from H_1 to H_6) $X' = X \cap E$ (resp. $X'' = X \cap \neg E$) the subset of worlds where the cause X appears simultaneously with (resp. without) coughing fits. We have:
$E = H_1 \cup H_2' \cup H_3' \cup H_4' \cup H_5$ and $H_1' = H_1$, $H_5' = H_5$, $H_6'' = H_6$.
According to the above knowledge, the physician can build a preference relation over the worlds and he is then endowed with the following order:
$H_6 < H_1, H_2', H_3 < H_2'', H_4', H_5 < H_4''$
It can be represented by concentric spheres around K in figure 2:

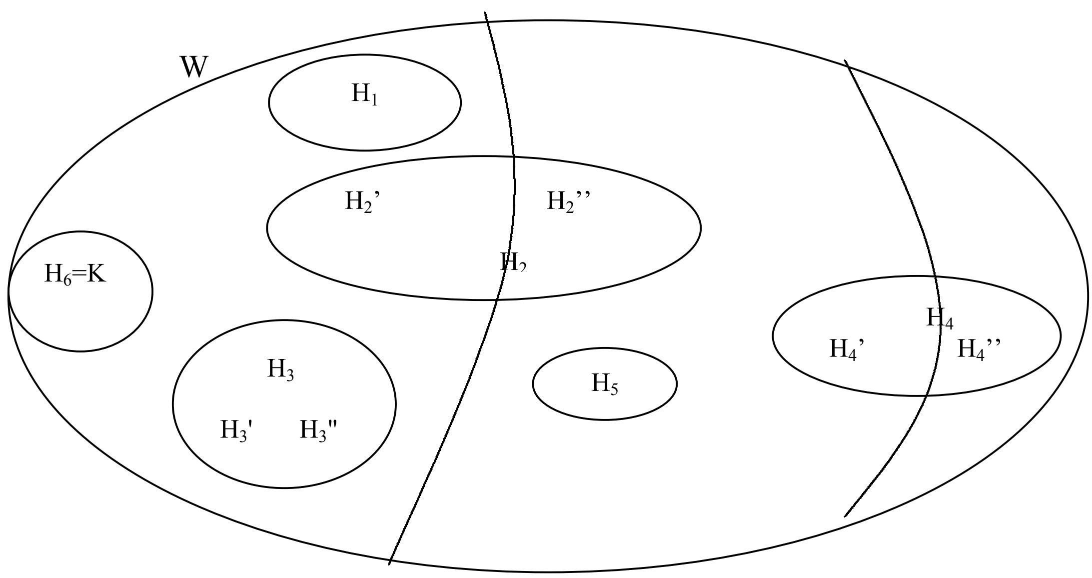

Figure 2

When the physician sees a patient, his initial belief K coincides with H_6 before having examined him. Indeed, he knows that most often the patient has nothing and just comes for a routine visit. However, when he learns that the patient is coughing, he has to revise his belief in $K*E= H_1 \cup H_2' \cup H_3'$. More generally, the rules of belief revision give:
$K*H_1=H_1$, $K*H_2=H_2'$, $K*H_3=H_3$, $K*H_4=H_4'$, $K*H_5=H_5$.

It is now easy to construct the following table indicating what the physician can abduce from fits of coughing with different notions of abduction:

	H_1	H_2	H_3	H_4	H_5
classical	Yes	No	No	No	Yes
non transitive	Yes	Yes	No	Yes	Yes
non reflexive	Yes	No	No	No	No
ordered	Yes	Yes	No	No	No

Using classical abduction, the physician abduces naturally bronchitis but he rejects asthma though it seems relevant also, even if it is not always accompanied of coughing fits; conversely, he accepts simulation of cough which is not very relevant in the context of a first visit. Using non transitive abduction, he accepts now asthma but he still accepts simulation of cough and even the sickness of the farmer's lung, a very strange diagnosis for an urban child. Using non reflexive abduction, he rejects simulation of cough as well as the sickness of the farmer's lung but he rejects asthma too. He keeps only bronchitis which provokes fits of coughing in all circumstances. Finally, using ordered abduction, he accepts the relevant diseases, i.e. bronchitis and asthma; conversely, he rejects all abnormal diseases such as simulation of cough or the sickness of the farmer's lung. It must be noted that an effort, even if it may give rise to coughing fits, is never abduced since cough may happen or not in usual cases.

4.2. General discussion

Classical abduction is inadequate for two reasons. It is too weak because a fact can be deduced from a lot of "strange" hypotheses since any subset of the antecedent set of worlds is an abduced consequent set of worlds. But all sufficient conditions can't be considered as "good explanations" of a derived fact. For instance, if I see something flying in the sky, I can abduce - but in a strange way - that it is a flying saucer since a flying saucer always flies. It is also too strict because a good explanation of a fact is not always a hypothesis from which this fact can be logically derived. In a lot of situations, no interesting deductive explanation (by sufficient conditions) may be available. For instance, if I see something flying in the sky, I cannot abduce - contrary to intuition - that this is a bird because if many birds fly, not all birds fly (penguins, ostriches).

Non transitive abduction takes into consideration the fact that deductive explanations are not always available and that most good explanations are often non monotonic inferences that can be defeated by counterexamples. It addresses correctly the second default of the classical abduction scheme, by accepting some good candidates that classical abduction would have rejected. For instance, it allows the abduction that some flying object in the sky could be a bird because normally a bird flies. But, it does not address its first default: it is still too weak and would lead to accept a lot of bad candidates for abduction. In particular, it doesn't discard the abduction about the flying saucer.

Non reflexive abduction and ordered abduction need a more precise discussion. First, it may be observed that they both satisfy the following preliminary condition: when receiving a new piece of information E, it is necessary first to revise the initial belief K according to message E before proceeding to abduction. Hence, both abductions (contrary to classical abduction and to non transitive abduction) concentrate on the best explanation of a fact by ruling out "abnormal" hypotheses, hypotheses that would explain E but not through the normal part of E.

An argument in favour of non reflexive abduction can be stated first. Consider a couple of events (H, E) such that $K * H \subseteq K * E$ but $H \not\subset K * E$. It is for instance the case in the example of § 4.1 when H is the hypothesis that the patient has asthma. It is possible to abduce H by ordered abduction but not by non reflexive abduction. Let's call H' the hypothesis K*H. It is possible to abduce H' by non reflexive abduction (and of course by ordered abduction too). Now, why should an agent abduce H in so far as he can abduce H' which seems to be a better explanation, in the sense that E is deductively implied by H'? One could think that non reflexive abduction which allows the agent to abduce H' and not H is a better type of abduction than ordered abduction which allows him to abduce H, since H is a not so good as an explanation than H'. For instance, if I see a flying object in the sky, the hypothesis that it is a "flying bird" (a "non penguin" bird) could be considered as a better abduction than the hypothesis that it is just a bird (which is not selected through non reflexive abduction).

However, this argument is not really relevant. It does not consider seriously enough the relevance of non monotonicity for ordinary (and even scientific) reasoning. The starting point of non monotonic logic is that the set of possible worlds handled by a reasoning agent is generally not refined enough to establish deductive relations between empirical events. The proposition "if A then B" is generally relative to a set of empirical conditions or "provisos" and the set of these provisos is generally computationally intractable or even infinite (Hempel, 1988). For instance (Goodman, 1955), if you see a lighted match, you can explain it by the fact that somebody scratched it, but it is not enough because you have also to assume that the match was not wet, that there was no wind and so on. Hence, ordinary reasoning is better represented by propositions such as "if A then normally B". The set of possible worlds considered by the modeler to give a semantic interpretation to this kind of propositions (in terms of "minimal worlds") is necessary finer than the set of possible worlds considered by the agent. Hence, it is a philosophical fallacy to recommend that the agent should use this finer set of worlds to perform his reasoning task. The proposition H'= K*H (the normal hypothesis with all its provisos) will generally not be expressible in the vocabulary used by the agent who is constrained to use H (the general hypothesis alone).

Incidentally, the standard "bird" example (like all examples in "small worlds") is a bit misleading because it is too simple. Speaking of "flying birds" treats H' as the conjunction of H and E. It is true that if a hypothesis H is a non monotonic explanation of E: $H \mid\sim E$ (or equivalently $K*H \subseteq E$), then the conjunction of E and H will be a deductive explanation of E (this is even true for any hypothesis H compatible with E). But it is not in the spirit of abduction to abduce from E the conjunction of E and of another hypothesis H. In the scientific work as well as in the usual life, due to the limitations of language, it is generally impossible to express a hypothesis which actually entails the observed event from a purely deductive point of view. By requiring that the hypothesis should deductively imply the normal cases of the fact, non reflexive abduction prevents from considering non monotonic relations between an explanans and an explanandum, and it can often be impossible to find an interesting hypothesis which satisfies this requirement.

Ordered abduction will then be favored as the most realistic type of abduction. It validates the idea that an explanation may be a non monotonic relation between hypotheses and facts, but conversely accepts the restriction that good explanations of an event are those which validate only its normal ways to be true, i.e. its preferred interpretations. It simultaneously allows the "bird" hypothesis and rules out the flying saucer one. This seems to a be a good compromise between the two defaults of classical abduction. An interesting consequence of this conclusion is that abduction cannot be simply defined by the inversion of a consequence relation which would describe "good explanations": neither deduction nor non monotonic inference are adequate definitions of good explanations.

Nevertheless, it is possible to lessen the gap between ordered and non reflexive abduction if one accepts to consider that, in a typical abduction situation, an agent would only hesitate between a fixed set of exclusive abducible hypotheses. These exclusive hypotheses are for instance the set of possible answers to one question (Levi, 1979), the possible diseases of a patient or the possible murderers for a crime (like in the game of *Cluedo*). Hence, the agent does not consider all possible subsets of the set of possible worlds W but the cells of a partition of W, belonging to $W' \subset 2^W$. From the agent's point of view, the reasoning task is performed within W' where any hypothesis is reduced to a single world. In that case, the definitions of ordered and of non reflexive abductions collapse since $K*H = H$ for any hypothesis H. Such a situation is in accordance with the previous remark: the set of possible hypotheses within which the abductive task is *de facto* performed is usually not enough refined to allow the agent to proceed to deductive explanations of an empirical phenomenon.

Remark: abduction is a dynamical reasoning, in the spirit of the Percean theory of $abduction_2$. Abduced hypotheses have to be tested before being adopted. One of them may be selected after this testing as the "best explanation". But all of them may be discarded. This would imply a revision of the context K in which abduction takes place and may lead to reactivate some hypotheses previously ignored. In the example of § 4.1, the physician will wait the results of some complementary analysis in order to fix his diagnosis between the best explanations of the cough. This analysis could confirm that the child suffers from asthma or from bronchia. But if none of these hypotheses is confirmed he may reconsider the possibility of malingering or of the sickness of the farmer's lung.

4.3. Related works

We will consider in this section the works which are directly related to the present paper i.e. which aim at formulating purely logical definitions of abduction either through a semantic criterion or through a list of axioms (or both). In the conclusion (§ 7), a brief comment will be made about the link of these works with other kinds of approaches, especially logic programming.

Pino-Pérez and Uzcátegui (1999) is the most recent and complete work about the links between different notions of abduction, non monotonic inference and belief revision. We will refer to this work in the hereafter comments.

Classical abduction can be associated with the axiom system proposed by Flach (1996) under the name of "explanatory induction", as shown by Pino-Pérez and Uzcátegui (1999, section 5).

Non transitive abduction is proposed by Boutilier and Becher (1995) under the name of "predictive explanation". It is introduced by Pino-Pérez and Uzcátegui (1999) under the label "epistemic explanation" in relation with belief revision.

Non reflexive abduction gives a belief revision semantics to the criterion proposed by Cialdea Mayer and Pirri (1996). It is introduced by Pino-Pérez and Uzcátegui (1999) under the label "causal explanation" in relation with non monotonic inference. The heuristic they adopt consists in relating abduction to non monotonic reasoning in the same spirit that we relate abduction to belief revision. More precisely, they associate to abduction (denoted $E \triangleright H$) an inference relation (denoted $E \mid\sim_{ab} F$) defined by:

$E \mid\sim_{ab} F$ if (if $E \triangleright H$ then $H \subseteq F$)

They impose to $\mid\sim_{ab}$ to satisfy several axioms of the non monotonic inference of Kraus, Lehmann & Magidor (1990) and they look for the corresponding axioms for $\triangleright$. They define stronger and stronger axiom systems with more axioms till reaching causal explanation with all axioms. The last system is shown to satisfy:

$E \triangleright H$ iff (if $E \mid\sim_{ab} F$ then $H \subseteq F$)

It is easy to see that it corresponds precisely to non reflexive abduction.

Ordered abduction is also considered by Pino-Pérez and Uzcátegui (1999) under the label "strong epistemic explanation" in relation with belief revision. In fact, they discard it in favor of non reflexive abduction by using two types of arguments. First, they notice that in some cases, ordered explanations "are not even explanations", in the sense that the observation E may not follow deductively from the abduced hypothesis H. However, the present paper vindicates the idea that good explanations are not necessarily deductive and even, that they are generally not. Second, they follow their own heuristic described before. But they don't give strong arguments for it. In fact, the same kind of heuristic leads to ordered abduction if the inference relation $\mid\sim_{ab}$ is defined by:

$E \mid\sim_{ab} F$ if (if $E \triangleright H$ then $H \mid\sim F$)

where $\mid\sim$ satisfies the KLM axioms. Now, if one assumes that $\mid\sim_{ab}$ satisfies also the KLM axioms (i.e. $\mid\sim_{ab}$ is the same inference relation than $\mid\sim$) then the reverse relation is:

$E \triangleright H$ iff (if $E \mid\sim F$ then $H \mid\sim F$)

It's straightforward to see that this is equivalent to ordered abduction.

Several of the preceding authors give axioms for the abduction schemes they consider. These axioms are for some of them similar to the axioms we will present now. But they are generally necessary and not sufficient : no full representation theorem is proved (except for classical abduction by Flach 1996).

5. AXIOMS AND REPRESENTATION THEOREMS

Considering classical abduction, we refer the reader to the Flach (1996) axiom system and representation theorem.

5.1. Non transitive abduction

Since non transitive abduction has been defined by reverse non monotonic inference, the following list of axioms is obtained trough the reversal of the axioms of rational non monotonic inference:

B1. Reflexivity
If $H \neq \emptyset$ then $H \| \approx H$

B5. Right Or
If $(E \| \approx H) \wedge (E \| \approx G)$ then $E \| \approx G \cup H$

B9. Left And
If $(E \| \approx H) \wedge (F \| \approx H)$ then $E \cap F \| \approx H$

B10. Left Weakening
If $(E \| \approx H) \wedge (E \subseteq F)$ then $F \| \approx H$

B11. Rational Right Strengthening
If $(E \| \approx H) \wedge$ not$(-F \| \approx H)$ then $E \| \approx F \cap H$

B1 means that every non contradictory hypothesis can be abduced from itself. B5 states that the disjunction of two hypotheses abduced from an event is also abduced from this event while B9 states that one hypothesis abduced from two events is abduced from the conjunction of these events. B10 asserts that if a hypothesis is abduced from an event which implies another one, it is also abduced from the last one. Finally, B11 asserts that if from an event one abduces a hypothesis which is not abduced from the negation of another event, the conjunction of the hypothesis and of the second even can be abduced from the first event.

No original representation theorem will be given.

5.2. Non reflexive abduction

The proposed axioms are the following:

B0. Non contradiction
If $E \| \prec H$ then $H \neq \emptyset$

B1'. Pointwise Reflexivity
$w \| \!\prec w$

B2. Strong Left Or
If $(E \| \!\prec F) \wedge (G \| \!\prec H)$ then $(E \cup G) \| \!\prec F \vee (E \cup G) \| \!\prec H$

B3. Infra Classicality
If $E \| \!\prec H$ then $H \subseteq E$

B4. Right Strengthening
If $(E \| \!\prec H) \wedge (G \subseteq H)$ then $E \| \!\prec G$

B5. Right Or
If $(E \| \!\prec H) \wedge (E \| \!\prec G)$ then $E \| \!\prec G \cup H$

B6. Weak Monotony
If $(E \| \!\prec H) \wedge (H \subseteq F)$ then $E \cap F \| \!\prec H$

B7. Weak Cut
If $(E \| \!\prec G) \wedge (G \subseteq F) \wedge ((E \cap F) \| \!\prec H)$ then $E \| \!\prec H$

B0 says that a contradiction can never be abduced and B1' states that every non empty world is always self abduced. B2 says that if two hypotheses are respectively abduced from two events, then one of them at least is abduced from the disjunction of the events. B3 means that one abduces only hypotheses from which the event can be deduced. Concerning the conclusion side, B4 says that it is always possible to strengthen an abduced hypothesis and B5 that it is always possible to abduce the disjunction of two abduced hypotheses. Concerning the premise side, B6 means that it is always possible to add to the premises of an abduction any consequence of the hypothesis while B7, in the opposite, means that it is always possible to cut among the premises of an abduction on the condition that one of the premises or an antecedent of it can be abduced from another premise.

The corresponding **representation theorem** states:

Theorem 1. Let * be a revision function satisfying AGM axiom system **A** = {A1, A2, A3, A4, A5}, then an inference relation $\| \!\prec$ defined according to $(E \| \!\prec H) \equiv (\emptyset \neq H \subseteq K{*}E)$ respects the set of axioms $\mathbf{B_{NR}}$ = {B0, B1', B2, B3, B4, B5, B6, B7} and therefore it is a non reflexive abductive inference relation.

Conversely, let $\| \!\prec$ be a non reflexive inference relation satisfying the axiom system $\mathbf{B_{NR}}$ = {B0, B1', B2, B3, B4, B5, B6, B7}. Then the operation * defined by $K{*}E = \cup H: E \| \!\prec H$ (union of all events abduced from E) where K=K*T, respects the axiom system **A** = {A1, A2, A3, A4, A5} and therefore it is a revision function. Moreover, $(E \| \!\prec H) \equiv (\emptyset \neq H \subseteq K{*}E)$ and $K{*}E = \{w: E \| \!\prec w\}$.

The proof is given in appendix 1.

Remark: notice that in this case, K*E can be seen as the set of all events abduced from E.

5.3. Ordered abduction

The proposed axioms are the following:

B1. Reflexivity
If $H \neq \emptyset$ then $H \| \!\approx H$

B3'. Weak Infra Classicality
If $E \| \!\approx H$ then $E \cap H \neq \emptyset$

B4'. Weak Right Strengthening
If $(E \| \!\approx H) \wedge (\emptyset \neq G \subseteq H)$ then $E \| \!\approx G) \vee (E \cap (-G)) \| \!\approx E$

B5. Right Or
If $(E \| \!\approx H) \wedge (E \| \!\approx G)$ then $E \| \!\approx G \cup H$

B6. Weak Monotony
If $(E \| \!\approx H) \wedge (H \subseteq F)$ then $E \cap F \| \!\approx H$

B8. Transitivity
If $(E \| \!\approx F) \wedge (F \| \!\approx G)$ then $E \| \!\approx G$

B9. Left And
If $(E \| \!\approx H) \wedge (F \| \!\approx H)$ then $(E \cap F) \| \!\approx H$

B1 is a strengthening of B0, every hypothesis being here self abduced. B3' restricts Infra Classicality to the fact that abduced hypotheses are at least not contradictory with the event considered. B4' weakens B4 and states that either it is possible to strengthen an abduced hypothesis from a given premise or that premise can be abduced from the conjunction of itself and the negation of the strengthened hypothesis. B5 and B6 are as before. B8 states a classical transitivity property. Finally, B9 says that abduction is preserved by the conjunction of premises from which the same hypothesis can be abduced.

The corresponding **representation theorem** states:

Theorem 2. Let * be a revision function satisfying AGM axiom system **A** = {A1, A2, A3, A4, A5}, then an inference relation $\| \!\approx$ defined according to $(E \| \!\approx H) \equiv (\emptyset \neq K*H \subseteq K*E)$ respects the axiom system $\mathbf{Bo_R}$ = {B1, B3', B4', B5, B6, B8, B9} and therefore it is an ordered abductive inference relation.

Conversely, let $\| \!\approx$ be an ordered inference relation satisfying the axiom system $\mathbf{Bo_R}$= {B1, B3', B4', B5, B6, B8, B9}. Then the operation * defined by $K*E = \cap H: H \| \!\approx E$ (intersection of all events from which E can be abduced) and where K=K*T, respects the axiom system **A** = {A1, A2, A3, A4, A5}, and therefore it is a revision function. Moreover, (E‖≈ H) ≡ (∅ ≠ K*H ⊆ K*E) and K*E = {w: E‖≈ w}.

The proof is given in appendix 2.

Remark: notice that in this case, K*E can be seen as the common part of all events from which E can be abduced. This result can be considered as less intuitive than the result obtained for non-reflexive abduction (where K*E can be seen as the set of all events abduced from E). However, in the case of ordered abduction, K*E ⊆ ∪H: E ‖≈ H. Hence, one keep the result that the final belief validates all the abduced hypotheses though it is no more constrained to be equal to this set of hypotheses.

6. AXIOMATIC COMPARISON BETWEEN THE ABDUCTION SCHEMES

6.1. Summary of axioms

Table 2 shows the logical links between the three axiom systems, discarding classical abduction. The axioms entering in their definition are presented in bold characters. The derivation of other axioms is proved in the appendix.

	NON REFLEXIVE ABDUCTION	ORDERED ABDUCTION	NON TRANSITIVE ABDUCTION
B0: Non Contradiction	**Yes**	Yes	Yes
B1': Pointwise Reflexivity	**Yes**	Yes	Yes
B3’ Weak Infra classicality	Yes	**Yes**	Yes
B4’: Weak Right Strengthening	Yes	**Yes**	Yes
B5 Right Or	**Yes**	**Yes**	**Yes**
B6: Weak Monotony	**Yes**	**Yes**	Yes
B7: Weak Cut	**Yes**	Yes	Yes
B9: Left And	Yes	**Yes**	**Yes**
B1: Reflexivity	No	**Yes**	**Yes**
B3: Infra Classicality	**Yes**	No	No
B4: Right Strengthening	**Yes**	No	No
B8: Transitivity	Yes	**Yes**	No
B10 Left Weakening	No	No	**Yes**
B11 Rational Right Strengthening	No	No	**Yes**

Table 2

Remark: Ordered abduction is logically weaker than non reflexive abduction. However, the axioms of the former are not all weakened with respect to the latter (B1’ becomes stronger while B3 and B4 become weaker). One may wonder how this is possible. In fact, what matters is whether the transformation of axioms implies an increase or a decrease of the number of couples (E, H) such that $E \| \!\rightarrow H$. An axiom transformation is said to be ampliative (resp. restrictive) if more (resp. less) couples respect the new axiom. Any axiom has the form “if antecedent then consequent”, where antecedent and consequent contain one formula of type $E \| \!\rightarrow H$. It is easy to show the following:

- if consequent alone is weakened (resp. strengthened), the corresponding axiom is weakened (resp. strengthened) and ampliative (resp. restrictive) ;
- if antecedent alone is weakened (resp. strengthened), the corresponding axiom is strengthened (resp. weakened) and ampliative (resp. restrictive).

It can be checked that B1' is submitted to a weakening of the antecedent, while B3 and B4 are submitted to a weakening of the consequent, hence all three are ampliative as it should be.

6.2. Comparison of axioms

A first group of eight axioms is common to all abduction schemes.

A second group of three axioms differentiates non reflexive and ordered abduction (and is common to ordered abduction and to non transitive abduction). Reflexivity cannot be considered as a wishful axiom since nothing is gained if one abduces the fact that one wants to explain; however, it can be considered as some degenerated case which is not really harmful. Infra Classicality and Right Strengthening correspond to an ideal deductive explanation scheme but are too demanding for common reasoning since they rule out most of the relevant abductions performed. A good illustration against Right Strengthening is given by Cialdea Mayer and Pirri (1996): the fact that some spoon of sugar has been added in my coffee is a good explanation of the fact that my coffee is sweet enough; but the fact that some spoon of sugar and some spoon of salt have been added is no more a good explanation of that sweetness. Both axioms are responsible for rejecting relevant hypotheses. Hence their rejection is in favor of ordered abduction.

A third group of three axioms differentiates ordered and non transitive abduction (and is common to non reflexive and ordered abduction). Transitivity is an aimed property if one wants to proceed to successive abductions. Left Weakening and Rational Right Strengthening imply to abduce lots of hypotheses which are not sufficiently sorted out. They are responsible for accepting abnormal hypotheses. Hence their rejection is again in favor of ordered abduction.

This discussion leads to the following conclusion: non transitive abduction does not capture very well the intuitive properties of abduction. Non reflexive abduction is generally unreachable for the reasons already detailed but appears as a sort of ultimate aim. It could be seen as a limit case of ordered abduction (as classical abduction could be seen as a limit case of non reflexive abduction). Ordered abduction appears to obey the best combination of axioms. In fact, the only remaining objection to ordered abduction is that it satisfies Reflexivity. This objection is not an argument in favor of the other abduction schemes. It rather points out one limitation of the framework of belief revision: the notion of explanatory power is not embedded in the underlying preference relation on the set of possible worlds.

7. CONCLUSION

Four abduction schemes were studied in the paper. A semantic definition was first proposed for each one using belief revision operations. This leads to a first comparison based on an example and on more general semantic considerations. The method was to discard schemes that allow an agent to abduce hypotheses he should normally not abduce or that prevent him from abducing hypotheses that he could be willing to abduce. Two schemes (non reflexive and ordered abduction) were considered as serious candidates for representing the intuitive meaning of abduction. An axiomatic system was then provided for each of them and original representation theorems were proved. That leads to a second comparison based on the desirability of the respective axioms. The method was to single out axioms which are satisfied by one scheme and violated by another and to make an appraisal of their relevance. Ordered abduction was finally considered as the best definition of abduction. Non reflexive abduction is considered as a sort of limit case which cannot be really reached due to the impossibility of clarifying all the provisos needed to reach a real deductive inference.

Another direction of research would be to apply the ideas of the paper towards a more procedural and computational goal. This is precisely what abductive logic programming (ALP) intends to do. However, there are some important differences between the ALP framework and the belief revision one. One is that in ALP the observation E is generally consistent with initial belief which is only completed by the new observation, while in belief revision the interesting case is when E is a "surprising" observation contradicting initial belief[1]. Furthermore, this very active field of research is not exempt of a more fundamental questioning concerning its semantical interpretation. Quoting Kakas & Denecker (2001), “the definition of an abductive solution defines the formal correctness criterion for abductive reasoning, but does not address the question of how the ALP formalism should be interpreted […]. For example, how is negation in ALP to be understood? […] Another open question is the relationship to classical logic”. Hence, the two approaches should be thought as complementary appraisals of abductive reasoning but their precise links remain to be studied.

The paper is mainly oriented towards an epistemological and theoretical goal. It tries to make a link between abductive reasoning and other logical developments such as belief revision and non monotonic inference. As such, further works could make the analysis deeper by extending the preceding definitions as well as the axioms. An infinite number of possible worlds would allow the modeler to deal with a larger set of propositions. The extension to predicate logic instead of propositional logic would make easier the distinction between the laws and the facts from which they are abduced. At last, the extension to probability calculus would allow to build a bridge with diagnosis analysis often treated in a probabilistic framework.

APPENDIX 1: REPRESENTATION THEOREM FOR NON REFLEXIVE ABDUCTION

Derived propositions

We show below other properties respected by non reflexive abduction.

B0'. If no hypothesis can be abduced from an event, then this event is empty.
It comes by recurrence from B1' and B2. (It is not a formal proposition hence cannot be incorporated in the axiom system as one may wish in order to spare axioms B1’ and B2).

[1] We thank Mark Dennecker for having drawn our attention on this point.

B1". Weak Reflexivity: If $E \Vert\!\prec H$ then $H \Vert\!\prec H$

B6 with F=H gives $(E\cap H) \Vert\!\prec H$. By B3, if $E \Vert\!\prec H$ then $H\subseteq E$, hence $E\cap H=H$.

B8. Transitivity: If $(E \Vert\!\prec F) \wedge (F \Vert\!\prec G)$ then $E \Vert\!\prec G$

By B3 and B4.

B9. Left And: If $(E \Vert\!\prec H) \wedge (F \Vert\!\prec H)$ then $E\cap F \Vert\!\prec H$

From B3 and B6.

B46. Pointwise left strengthening: If $(E \Vert\!\prec H) \wedge \neg (E \Vert\!\prec w)$ then $E\cap(-w) \Vert\!\prec H$

If $(E \Vert\!\prec H) \wedge \neg (E \Vert\!\prec w)$ then $\neg(w\subseteq H)$; otherwise, by B4 $(E \Vert\!\prec H) \wedge (w\subseteq H)$ would give $E \Vert\!\prec w$. Hence $(E \Vert\!\prec H) \wedge H\subseteq(-w)$ and then $E\cap(-w) \Vert\!\prec H$ from B6.

B6'. If $(E \Vert\!\prec H) \wedge (E \Vert\!\prec F) \wedge (F\subseteq H)$ then $H \Vert\!\prec F$

By B6: $(E\cap H) \Vert\!\prec F$. By B3, $H\subseteq E$ hence $E\cap H=H$.

B26. If $(E \Vert\!\prec H) \wedge (G\subseteq E)$ then $H\cup G \Vert\!\prec H$

From B1" $H \Vert\!\prec H$ hence by B2 $E\cup H \Vert\!\prec H$. Now $(E\cup H \Vert\!\prec H) \wedge (H\subseteq H\cup G)$ and B6 give $(E\cup H) \cap (H\cup G) \Vert\!\prec H$. And $(E\cup H) \cap (H\cup G) = H\cup G$ if $(G\subseteq E)$.

B3". If $E \Vert\!\prec H$ then $E \Vert\!\prec E\cap H$

Trivial because from B3, $E\cap H=H$.

B12 .Weak Supra Classicality: if $(E \Vert\!\prec E) \wedge (E \subseteq H)$ then $E \Vert\!\prec H$

Trivial from B6.

B67. If $(E \Vert\!\approx H) \wedge (G\subseteq E) \wedge (H\cup G) \Vert\!\approx G$ then $E \Vert\!\approx G$

By B6 $(G\subseteq E) \wedge (H\cup G) \Vert\!\approx G$ gives $(H\cup G) \cap E \Vert\!\approx G$.

By B7 $(E \Vert\!\approx H) \wedge (H\subseteq H\cup G) \wedge (H\cup G) \cap E \Vert\!\approx G$ gives $E \Vert\!\approx G$.

Representation theorems

***Theorem 1*:**

Let * be a revision function satisfying AGM axiom system **A** = {A1, A2, A3, A4, A5}, then an inference relation $\Vert\!\prec$ defined according to $(E \Vert\!\prec H) \equiv (\varnothing \neq H \subseteq K{*}E)$ respects the set of axiom system $\mathbf{B_{NR}}$ = {B0, B1', B2, B3, B4, B5, B6, B7} and therefore it is a non reflexive abductive inference relation.

Proof**: (**We will use equally $E \Vert\!\prec H$ or $H \subseteq K{*}E$ with $\varnothing \neq H$)

B0: trivial by definition.

B1': trivial because for every world w, K*w=w.

B2: let $E \| \!\prec F$ and $H \| \!\prec G$ i.e. $F \subseteq K^*E$ and $G \subseteq K^*H$. From A45: $K^*(E \cup H) = K^*E$ or K^*H or $(K^*E \cup K^*H)$. Hence $F \subseteq K^*(E \cup H)$ or $G \subseteq K^*(E \cup H)$ hence $E \cup H \| \!\prec F$ or $E \cup H \| \!\prec G$.

B3: If $H \subseteq K^*E$ then $H \subseteq E$ because $K^*E \subseteq E$ by A2.

B4: trivial.

B5: If $E \| \!\prec H$ and $E \| \!\prec G$ then $H \subseteq K^*E$ and $G \subseteq K^*E$. Then $G \cup H \subseteq K^*E$ hence $E \| \!\prec G \cup H$.

B6: Assume $\varnothing \neq H \subseteq K^*E$ and $H \subseteq F$. Then $H \subseteq K^*E \cap F$. By A4: $K^*E \cap F \subseteq K(^*E \cap F)$. Hence $H \subseteq K^*(E \cap F)$.

B7: Assume $G \subseteq K^*E$, $G \subseteq F$, $H \subseteq K^*(E \cap F)$. By A5, $K^*(E \cap F) \subseteq K^*E \cap F$. Hence $H \subseteq K^*E$.

Theorem 2:

Let $\| \!\prec$ be a non reflexive inference relation satisfying the axiom system $\mathbf{B}_{\mathbf{NR}}$ = {B0, B1', B2, B3, B4, B5, B6, B7}. Then the operation * defined by $K^*E = \cup H, E \| \!\prec H$ (union of all events abduced from E) where we set K=K*T, respects the axiom system **A**={A1, A2, A3, A4, A5} and therefore it is a revision function. Moreover, $(E \| \!\prec H) \equiv (\varnothing \neq H \subseteq K^*E)$ and $K^*E = \{w: E \| \!\prec w\}$.

Proof:

a) We show first that $(E \| \!\prec H) \equiv (\varnothing \neq H \subseteq K^*E)$.

If sense: If $\varnothing \neq H \subseteq K^*E$ then $E \| \!\prec H$.

Let Abd(E) be the set of events abduced from E. By B5, Abd(E) is closed under union. By B4, Abd(E) is closed under the sub-set operation.

Let $\varnothing \neq H \subseteq K^*E$. There exists a family $\{F_i\}$ of elements from Abd(E) such as $H \subseteq \cup F_i$. Now $\cup F_i \in$ Abd(E) and since Abd(E) is closed under sub-set operation $H \in$ Abd(E) hence $E \| \!\prec H$.

Only if sense: If $E \| \!\prec H$ then $\varnothing \neq H \subseteq K^*E$.

Trivial from the definition of K*E and B0.

b) Let's show now that $K^*E = \{w, E \| \!\prec w\}$.

Let w be abduced from E. Then $\{w\} \subseteq K^*E$ hence $w \in K^*E$. Vice versa, let $w \in K^*E$, hence there exist H such as $E \| \!\prec H$ and $\{w\} \subseteq H$ hence by B4, $E \| \!\prec \{w\}$.

c) We can now prove that * is a revision function satisfying the axioms A1 to A5:

A1: Assume $E \neq \varnothing$. If E is a single world then $E \| \!\prec E$ and $K^*E = E \neq \varnothing$. If E contains more than a world, let $E = \cup w_i$, $i \in I$ with $I = \{1, 2, \ldots.\}$. Now, $w_i \| \!\prec w_i$ for every i by B1'.

Then $w_1 \cup w_2 \| \prec w_1$ or $w_1 \cup w_2 \| \prec w_2$ by B2. Assume now that $\cup w_i \| \prec w_\alpha$ for i, $\alpha \in I' \subset I$. Let $j \in I-I'$. B2 gives: $(\cup w_i) \cup w_j \| \prec w_\alpha$ or $(\cup w_i) \cup w_j \| \prec w_j$. By recurrence, there exists some $\beta \in I$ such that $E \| \prec w_\beta$ hence $K*E \neq \varnothing$. Moreover this proves that in every case $K \neq \varnothing$ because $K=K*T$ and $T \neq \varnothing$.

A2: trivial by B3.

A3: Assume $K \subseteq E$ then $K*T \subseteq E$. Let's show that $K*E=K=K*T$.
a) Let $H \subseteq K*T$ then $T \| \prec H$ and $H \subseteq E$. By B6, $E \| \prec H$ then $H \subseteq K*E$. Then $K*T \subseteq K*E$.
b) Let $H \subseteq K*E$. By A1, it exists $F \neq \varnothing$ such as $T \| \prec F$. Then from a) $F \subseteq E$. Then $T \| \prec F$ and $F \subseteq E$ and $E \| \prec H$. By B7, $T \| \prec H$ hence $H \subseteq K*T$. Then $K*E \subseteq K*T$.
*(Remark: This proof is unnecessary if we adopt the equivalent axiom system {A1, A2, A4, A5, K*T=K} for revision.)*

A4: Let $H \subseteq (K*E) \cap F$. Then $E \| \prec H$ and $H \subseteq F$. Then by B6, $E \cap F \| \prec H$ hence $H \subseteq K*(E \cap F)$.

A5: Assume $(K*E) \cap F \neq \varnothing$. Then it exists G such as $E \| \prec G$ and $G \subseteq F$. By A1, $K*(E \cap F) \neq \varnothing$ because $(E \cap F) \neq \varnothing$ since $(K*E) \cap F \neq \varnothing$ and $K*E \subseteq E$. So let $H \subseteq K*(E \cap F)$ i.e. $E \cap F \| \prec H$. By B7, $E \| \prec H$ then $H \subseteq (K*E)$. But as $E \cap F \| \prec H$, $H \subseteq F$ by B3. Hence $H \subseteq (K*E) \cap F$.

APPENDIX 2: REPRESENTATION THEOREM FOR ORDERED ABDUCTION

Derived propositions
We show below other properties respected by ordered abduction.

B0. Non contradiction: If $E \| \approx H$ then $H \neq \varnothing$
Trivial from B3'.

B14. Reflexive Weak Right Strengthening: If $(G \subseteq E) \wedge (G \neq \varnothing)$ then $(E \| \approx G) \vee (E \cap (-G) \| \approx E)$.
From B4 with E=H and B1. Moreover we can't have $E \| \approx G$ and $(E \cap (-G)) \| \approx E$; otherwise by B8 we would have $E \cap (-G) \| \approx G$ which is contradictory with B3'.

B2. Strong Left Or: If $(E \| \approx F) \wedge (G \| \approx H)$ then $(E \cup G \| \approx F) \vee (E \cup G \| \approx H)$.
B14 with $E \subseteq E \cup G$ and $G \subseteq E \cup G$ proves that if neither $E \cup G \| \approx E$ nor $E \cup G \| \approx G$ then $G \cap (-E) \| \approx E \cup G$ and $E \cap (-G) \| \approx E \cup G$. Hence by B9 a contradiction with B3'. Then $E \cup G \| \approx E$ or $E \cup G \| \approx G$. Hence B2 through B8 .

B2'. Left Or: If $(E \| \approx F) \wedge (G \| \approx F)$ then $E \cup G \| \approx F$
Trivial from B2.

B26. If $(E \|\approx H) \wedge (G \subseteq E)$ then $H \cup G \|\approx H$

From B1 $H \|\approx H$ hence by B2' $E \cup H \|\approx H$. Now $(E \cup H \|\approx H) \wedge (H \subseteq H \cup G)$ and B6 give $(E \cup H) \cap (H \cup G) \|\approx H$. And $(E \cup H) \cap (H \cup G) = H \cup G$ if $G \subseteq E$.

B10. If $(E \|\approx H) \wedge (G \subseteq E) \wedge \neg(E \|\approx G)$ then $E \cap (-G) \|\approx H$

From B14 and B8

B7. Weak Cut: If $(E \|\approx G) \wedge (G \subseteq F) \wedge (E \cap F \|\approx H)$ then $E \|\approx H$

Assume that $(E \|\approx G) \wedge (E \cap F \subseteq E) \wedge \neg(E \|\approx E \cap F)$. Then by B10 $E \cap (-E \cup -F) \|\approx G$ i.e. $E \cap (-F) \|\approx G$. This is contradictory with $G \subseteq F$ by B3'. Hence $(E \|\approx G) \wedge (G \subseteq F)$ gives $E \|\approx E \cap F$. Then by B8, $(E \|\approx G) \wedge (G \subseteq F) \wedge (E \cap F \|\approx H)$ gives $E \|\approx H$.

B3''. If $E \|\approx H$ then $E \|\approx E \cap H$

Assume that $E \|\approx E \cap H$ is not the case. Then by B10 $E \cap (-E \cap H) \|\approx H$ i.e. $E \cap (-H) \|\approx H$. Hence a contradiction by B3'.

B67. If $(E \|\approx H) \wedge (G \subseteq E) \wedge (H \cup G) \|\approx G$ then $E \|\approx G$

By B6 $(G \subseteq E) \wedge (H \cup G) \|\approx G$ gives $(H \cup G) \cap E \|\approx G$.

By B7 $(E \|\approx H) \wedge (H \subseteq H \cup G) \wedge (H \cup G) \cap E \|\approx G$ gives $E \|\approx G$.

Now, we show the equivalence between two axiom systems, the second containing less axioms than the first

Theorem:

The set of axioms $\mathbf{B_{OR}}$ = {B1, B3', B4', B5, B6, B8, B9} and $\mathbf{B'_R}$ = {B3', B14, B5, B6, B8, B9} are equivalent.

Proof:

It suffices to prove that under the other axioms, B14 is equivalent to the conjunction of B1 and B4'.

We have already proved that B14 follows from the conjunction of B1 and B4'.

Conversely, assume B14. B1 follows immediately under B3' if we set E=G. Let's show that B4' follows equally. Assume that $(E \|\approx H) \wedge (\emptyset \neq G \subseteq H)$. Now by B14, from $(\emptyset \neq G \subseteq H)$, it follows that $(H \|\approx G) \vee (H \cap (-G) \|\approx H)$. If $H \|\approx G$ then by B8, $E \|\approx G$.

So, to complete the proof, it suffices to show that: if $(E \|\approx H) \wedge (\emptyset \neq G \subseteq H) \wedge (H \cap (-G) \|\approx H)$ then $E \cap (-G)] \|\approx E$. In this case, we have not $H \|\approx G$ (see the proof above).

a) If $G \subseteq (-E)$, then $E \cap (-G) = E$. Hence by B1, $E \cap (-G)\ ) \|\approx E$

b) If $G \subseteq E$, then from B14 it follows that $(E \|\approx G) \vee (E \cap (-G) \|\approx E)$. Let's show that we have not $E \|\approx G$. If we assume the opposite, then we have $(E \|\approx G) \wedge (G \subseteq H) \wedge (G \subseteq E)$. Then $G \subseteq E \cap H$. Then from B14, it follows that either $E \cap H \|\approx G$ or $G \cap (-(E \cap H)) \|\approx (E \cap H)$. The latter case is impossible since $G \cap (-(E \cap H)) = \emptyset$. So $E \cap H \|\approx G$. Now by B14, from $E \cap H \subseteq H$

it follows that either $H \|\!\approx E\cap H$ or $H\cap(-(E\cap H)) \|\!\approx H$. The latter case is impossible because $H\cap(-(E\cap H))=H\cap(-(E))$; so we would have $H\cap(-(E)) \|\!\approx H$ which is contradictory to $E \|\!\approx H$ under B9. So $H \|\!\approx E\cap H$. Then by B8, $H \|\!\approx G$ in contradiction with the hypothesis. So we have not $E \|\!\approx G$. Hence we have $E\cap(-G) \|\!\approx E$.

c) In general $G=G_1\cup G_2$ with $G_1\subseteq E$ and $G_2\subseteq(-E)$. So $E\cap(-G)= E\cap(-G_1)$. So it suffices to prove that the conditions respected by G are respected by G_1 and to use the proof b). The only point to show is that if $(G_1\cup G_2\subseteq H)$ and if we have not $(H \|\!\approx G_1\cup G_2)$ then we don't have $H \|\!\approx G_1$. Or, what is equivalent, if $(G_1\cup G_2\subseteq H)$ and $H \|\!\approx G_1$ then $H \|\!\approx G_1\cup G_2$. Now by B1, $H\cap(G_1\cup G_2)=(G_1\cup G_2) \|\!\approx G_1\cup G_2$. Then from B7 (we can use it as it follows from other axioms than B4'): if $(H \|\!\approx G_1)\wedge(G_1\subseteq G_1\cup G_2)\wedge(H\cap(G_1\cup G_2)) \|\!\approx (G_1\cup G_2)$ then $H) \|\!\approx (G_1\cup G_2)$.

Representation theorem

Theorem 1:

Let * be a revision function satisfying AGM axiom system **A** = {A1, A2, A3, A4, A5}, then an inference relation $\|\!\approx$ defined according to $(E \|\!\approx H) \equiv (\emptyset \neq K{*}H \subseteq K{*}E)$ respects the axiom system $\mathbf{B_{OR}}$ = {B1, B3', B4', B5, B6, B8, B9} and therefore it is a reflexive abductive inference relation.

Proof:

B1: Trivial

B3': Let $(E \|\!\approx H)$ then $\emptyset\neq K{*}H\subseteq K{*}E$. Then by A2 $K{*}H\subseteq H$ and $K{*}E\subseteq E$. Hence $K{*}H\subseteq E\cap H\neq\emptyset$.

B4': Let $(E \|\!\approx H)$ and $(G\subseteq H)$.

Assume first that $G\cap K{*}H\neq\emptyset$. As $G=(G\cap H)$, $K{*}G=K{*}(G\cap H)$. Hence by A4 and A5 $K{*}G=G\cap K{*}H\subseteq K{*}H$. Now $K{*}H\subseteq K{*}E$ hence $K{*}G\subseteq K{*}E$, i.e. $E \|\!\approx G$.

Assume now that $G\cap K{*}H=\emptyset$. Now $K{*}H\subseteq K{*}E\subseteq E$ and $K{*}H\subseteq H$ by A2. So $K{*}E\cap H\neq\emptyset$. Hence $K{*}H=K{*}H\cap E=K{*}(E\cap H)=K{*}E\cap H$. Then $G\cap K{*}H=\emptyset$ gives $G\cap K{*}E\cap H=\emptyset$ then $G\cap K{*}E=\emptyset$ i.e. $K{*}E\subseteq -G$. Then $K{*}E\cap(-G)=K{*}E\neq\emptyset$. Then by A4 and A5, $K{*}(E\cap(-G))= K{*}E\cap(-G)=K{*}E$. Hence $E\cap(-G) \|\!\prec E$.

B5: Let $(E \|\!\approx F) \wedge (E \|\!\approx H)$ then $(K{*}F\subseteq K{*}E) \wedge (K{*}H\subseteq K{*}E)$. A2, A4 et A5 gives A45 (Right Distributivity) then $K{*}(F\cup H)$ is equal to either $K{*}F$ or $K{*}H$ or $K{*}F\cup K{*}H$. Hence $K{*}(F\cup H) \subseteq K{*}E$. Hence $E \|\!\approx (F\cup H)$.

B6: Let $(E \|\!\approx H) \wedge (H\subseteq F)$ then $(K{*}H\subseteq K{*}E) \wedge (H\subseteq F)$. By A2, $K{*}H\subseteq H$. Then $K{*}H\subseteq F$. By A4 $K{*}H\subseteq K{*}E\cap F\subseteq K{*}(E\cap F)$. Hence $E\cap F \|\!\approx H$.

B8: Let $(E \|\!\approx F) \wedge (F \|\!\approx G)$ then $K{*}F\subseteq K{*}E$ and $K{*}G\subseteq K{*}F$ then $K{*}G\subseteq K{*}E$ hence $E \|\!\approx G$.

B9: Let $(E\|\approx H) \wedge (F\|\approx H)$ then $K^*H\subseteq K^*E$ et $K^*H\subseteq K^*F$. By A2, $K^*F\subseteq F$ then $K^*H\subseteq K^*E\cap F$. Hence by A4, $K^*H\subseteq K^*(E\cap F)$ then $(E\cap F)\ \|\approx\ H$.

Theorem 2:

Let $\|\approx$ be a reflexive inference relation satisfying the axiom system $\mathbf{B_{OR}} = \{B1, B3', B4', B5, B6, B8, B9\}$. Then the operation * defined by $K^*E = \cap H\!: H\|\approx E$ (intersection of all events from which E can be abduced) and where we set $K=K^*T$, respects the axiom system $\mathbf{A} = \{A1, A2, A3, A4, A5\}$ and therefore it is a revision function. Moreover, $(E\|\approx H) \equiv (\emptyset \neq K^*H \subseteq K^*E)$ and $K^*E = \{w\!: E\|\approx w\}$.

corollary:
If $G\subseteq K^*E$ then $K^*G=G$.
Proof: It's enough to show that $G\subseteq K^*G$ (the other direction comes from B2). Let's show that if $G\cap(-K^*G)\neq\emptyset$ then $G\|\approx G\cap(-K^*G)$ which is contradictory as it means $K^*[G\cap(-K^*G)]\subseteq K^*G$ when by A2 $K^*[G\cap(-K^*G)]\subseteq G\cap(-K^*G)$.
By B14, $\emptyset\neq[G\cap(-K^*G)]\subseteq E$ implies either $E\|\approx G\cap(-K^*G)$ or $E\cap[-(G\cap(-K^*G))]\ \|\approx E$. In this latter case, $[E-(G\cap(-K^*G))]\ \|\approx E$. Then, $K^*E\subseteq K^*[E-(G\cap(-K^*G))]$ hence by A2, $K^*E\subseteq[E-(G\cap(-K^*G))]$ which is contradictory because $G\cap(-K^*G)\subseteq K^*E$. Hence $E\|\approx G\cap(-K^*G)$.
By B6, $E\|\approx G\cap(-K^*G)$ and $G\cap(-K^*G)\subseteq G$ imply $(E\cap G)\|\approx G\cap(-K^*G)$ then $G\|\approx G\cap(-K^*G)$. As we have shown that it is contradictory, then $G\cap(-K^*G)=\emptyset$. QED.

Proof of the theorem**:**

a) We show first that $(E\|\approx H)\ ) \equiv\ (\emptyset \neq K^*H\subseteq K^*E)$
If sense: if $(\emptyset \neq K^*H\subseteq K^*E)$ then $E\|\approx H$.
Let $K^*H\subseteq K^*E$ hence if $F\|\approx E$ then $K^*H\subseteq F$. Then $K^*H\subseteq E$ because $E\|\approx E$. Then by B14, $E\|\approx K^*H$ or $E\cap(-K^*H)\ \|\approx E$. But if $E\cap(-K^*H)\ \|\approx E$ then $K^*H\subseteq E\cap(-K^*H)$ which is impossible. Then $E\|\approx K^*H$. Now $K^*H\|\approx H$ by B9 so $E\|\approx H$ by B8.
Only if sense: If $E\|\approx H$ then $(\emptyset \neq K^*H\subseteq K^*E)$.
$K^*H= \cap G/G\|\approx H$ and $K^*E= \cap F\!:\!F\|\approx E$. Assume $E\|\approx H$. By B8, if $F\|\approx E$ then $F\|\approx H$. Hence $\{F\!:\!F\|\approx E\ \}\subseteq \{G\!:\!G\|\approx H\}$. Then $[\cap G\!:\!G\|\approx H] \subseteq [\cap F\!:\!F\|\approx E]$ hence $K^*H\subseteq K^*E$.
Now, by B9, $[\cap G\!:\!G\|\approx H]\ \|\approx H$ then by B3', $K^*H\cap H\neq\emptyset$

b) Let's show now that $K^*E = \{w\!: E\|\approx w\}$.
Let $w\ /\ E\|\approx w$ then $w\subseteq[\cap H\!: H\|\approx E]$. Indeed, $w\subseteq[\cap H\!: H\|\approx E]$ is equivalent to (if $H\|\approx E$ then $w\subseteq H$). Now $H\|\approx E$ et $E\|\approx w$ imply $H\|\approx w$ by B8. Then $H\cap w\neq\emptyset$ by B3' hence $w\subseteq H$.

Conversely, let w such as if $H \| \approx E$ then $w \subseteq H$. Then $w \subseteq E$ because $E \| \approx E$. Assume $E \| \approx w$ is not the case. Then by B10, from $(E \| \approx E) \wedge (w \subseteq E) \wedge \neg(E \| \approx w)$, one obtains $E \cap (-w) \| \approx E$. Now $w \subseteq (E \cap (-w))$ is not the case and this is in contradiction with [if $H \| \approx E$ then $w \subseteq H$].

c)We can now prove that the axioms are satisfied. Since by definition $K^*T=K$, it's enough to show that {A1, A2, A4, A5} is respected, by using the axiom system equivalent to **A**.

A1. By B1, $\emptyset \neq E \| \approx E$. So there exists at least one H such as $H \| \approx E$. By B9, $[\cap H: H \| \approx E] \| \approx E$ i.e. $K^*E \| \approx E$. By B3', $K^*E \cap E \neq \emptyset$. The same reasoning with $E=T$ shows that $K=K^*T$ is never empty.

A2. By B1 $\emptyset \neq E \| \approx E$ then $[\cap H: H \| \approx E] \subseteq E$.

A4. B4 shows that If $(E \| \approx H) \wedge (H \subseteq F)$ then $E \cap F \| \approx H$, hence if $(K^*H \subseteq K^*E) \wedge (H \subseteq F)$ then $K^*H \subseteq K^*(E \cap F)$. Let $G \subseteq (K^*E) \cap F$. We have $K^*G \subseteq G \subseteq K^*E$ and $G \subseteq F$. Then $K^*G \subseteq K^*(E \cap F)$. Now $K^*G=G$ by the corollary. This shows that $(K^*E) \cap F \subseteq K^*(E \cap F)$.

A5. Assume that $((K^*E) \cap F \neq \emptyset)$ then $(K^*(E \cap F) \subseteq (K^*E) \cap F)$.
By B14, $(E \cap F) \subseteq E) \wedge ((E \cap F) \neq \emptyset)$ implies $E \| \approx (E \cap F)$ or $E \cap (-F) \| \approx E$. Then $K^*(E \cap F) \subseteq K^*E$ or $K^*E \subseteq K^*[E \cap (-F)]$. But by A2, $K^*[E \cap (-F)] \subseteq [E \cap (-F)]$ which is contradictory with $(K^*E) \cap F \neq \emptyset$. Then $K^*(E \cap F) \subseteq K^*E$. And by A2, $K^*(E \cap F) \subseteq E \cap F$.

ACKNOWLEDGMENTS

We thank Paul Bourgine for its contribution to the global taxonomy, David Makinson and Jerome Lang for several constructive suggestions, the participants in seminars in Paris (cognitive economics) and Torino (LOFT 5 Conference) and two anonymous referees for helpful remarks.

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